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Delinearization of Quantum Logic

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The algebraic structure of the set of elementary observables of a delinearized quantal theory is described. As the delinearization procedure provides a kind of classical representation for any quantal theory, its relation to the traditional hypothesis of hidden variables is discussed.

1. INTRODUCTION

It is known that any order-unit normed space admits an isometric linear functional representation as a linear subspace of a space of real continuous functions over a compact set [the Kadison representation theorem (Alfsen, 1971; Asimov and Ellis, 1980). The question of a physical interpretation of this representation arises in a natural way if we take into account the fundamental position occupied by order-unit Banach spaces in any statistical physical theory in its operational (or "convex") generalization [see, e.g., Lahti and Bugajski (1985) and references therein]. Two such interpretations, closely related to one another, have been suggested recently. Namely the Kadison map can be seen as the delinearization of a quantal theory, i.e., as a formal description of the transformation from a quantal theory to its nonlinear extension (Bugajski, 1991). On the other hand, the dual Kadison map provides a statistical interpretation for quantal mixed states in terms of classical probability measures, so the Kadison representation can be placed among phase-space models of quantum theories (Bugajski, 1993; Busch *et al.*, 1993).

Once we have ascribed a physical meaning to the Kadison representation, we should explain how it is possible that we obtain a kind of classical embedding of a quantal theory in spite of the theorems forbidding "hidden variables" (Zierler and Schlesinger, 1965). One easily guesses that it is

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possible only at the price of destroying the elaborated structure of the quantum logic. In this way we come to the intriguing question: how does the quantum logic behave under the delinearization? The main goal of this paper is to report a preliminary study of algebraic aspects of the delinearization of quantum logic.

2. NONLINEAR EXTENSION OF QUANTUM LOGIC

Let W be an order-unit normed Banach space with e the order unit and o the origin. If W is one of the basic Banach spaces underlying an operational statistical theory, the order interval $[o, e]_W := \{a \in W \mid o \leq a \leq e\}$ is the set of effects of the theory. An observable related to W is a mapping $A: \mathbb{B}(\mathbb{R}) \to [o, e]_W$, where $\mathbb{B}(\mathbb{R})$ is the σ-field of Borel subsets of the real line $\mathbb{R}$, such that $A(\mathbb{R}) = e$, and for any sequence $X_1, X_2, \ldots$ of mutually disjoint Borel sets, $A(\bigcup X_n) = \sum A(X_n)$ with the right-hand side converging in the weak Banach topology of W. Thus, effects are "elementary observables." An observable is "sharp" if its range is contained in $\mathrm{Ex}[o, e]_W$, the set of extreme elements of $[o, e]_W$. It is not guaranteed in general that $\mathrm{Ex}[o, e]_W$ is not trivial; nevertheless, in the standard quantum mechanics as well as in its operational generalization [in both theories, $W = \mathscr{L}_s(\mathscr{H})$, the Banach space of bounded self-adjoint operators on a separable Hilbert space $\mathscr{H}$], $\mathrm{Ex}[o, e]_W$ is the set of all orthogonal projections on $\mathscr{H}$.

The functional representation of W relevant to the delinearization procedure is done by a map D of W into $C(\overline{\mathrm{Ex}S}^{W^*})$, the Banach space of real continuous functions on $\overline{\mathrm{Ex}S}^{W^*}$ with the sup-norm. S is the set of states of W, i.e., the base of the base normed Banach space W^*; $\mathrm{Ex}S$ is the set of extreme elements of S, topologized by the topology induced by the weak* topology of W^*; and $\overline{\mathrm{Ex}S}^{W^*}$, denoted in the sequel by Ω, is the weak* closure of $\mathrm{Ex}S$ in W^*. The delinearization map D (the Kadison map) is then defined by evaluation:

$$W \ni a \to Da \in C(\Omega), \qquad (Da)(\alpha) := \alpha(a), \qquad \alpha \in \Omega$$

The map D is a bipositive linear isometry between W and its image $D(W)$, De is the constant-1 function on Ω (Alfsen, 1971; Asimov and Ellis, 1980).

There are good reasons to believe that the quantum logic $\mathbb{L}$ is a subset of $\mathrm{Ex}[o, e]_W$. It is customary to assume $\mathbb{L} = \mathrm{Ex}[o, e]_W$, which is true in the case of theories with $W = \mathscr{L}_s(\mathscr{H})$. In the general case it seems that physical intuitions concerning quantum logic are better met by the set of projective units of W; Alfsen and Shultz (1976) give conditions under which the set of projective units coincides with $\mathrm{Ex}[o, e]_W$.

If we take as the starting point of our considerations an abstract quantum logic $\mathbb{L}$, then we cannot apply the delinearization procedure without embedding $\mathbb{L}$ in an order-unit space. Such an embedding can be constructed in many ways; we will follow Fischer and Rüttimann (1978). Let $(\mathbb{L}, \leq, e, \sim)$ be an orthomodular poset. We consider the space W' of all additive real functions on $\mathbb{L}$, $\alpha \in W'$ iff $a \leq \sim b$ implies $\alpha(a \vee b) = \alpha(a) + \alpha(b)$ for any $a, b \in \mathbb{L}$. The space W' is proved by Fischer and Rüttimann (1978) to be a base normed Banach space; its base S is the set of all states on $\mathbb{L}$, i.e., $\alpha \in S$ iff $\alpha : \mathbb{L} \to [0, 1]$ (the unit interval of $\mathbb{R}$) and α is additive on $\mathbb{L}$. We will assume that S is not empty. The Banach predual of W', denoted W, exists only if $S \neq \varnothing$ (Fischer and Rüttimann, 1978, Theorem 2). W is an order-unit normed Banach space, and $\mathbb{L}$ can be canonically identified with a subset of the unit interval of W: $L \ni a \to \phi a \in W$, $\alpha(\phi a) := \alpha(a)$ for any $\alpha \in W' = W^*$. Here ϕ is an injection if S separates elements of $\mathbb{L}$, and ϕe is the order unit of W; the conditions under which $\phi(\mathbb{L})$ is contained in $\mathrm{Ex}[\phi o, \phi e]_W$, where $o = \sim e$, are discussed in Cook (1978). The structure of $\mathbb{L}$ is preserved under ϕ if S is full on $\mathbb{L}$, i.e., if $\alpha(a) \leq \alpha(b)$ for all $\alpha \in S$ implies $a \leq b$, for any $a, b \in \mathbb{L}$. We will assume that S is full; hence we will identify $\mathbb{L}$ with its image under ϕ. Having $\mathbb{L}$ represented by effects of W, we can apply the delinearization procedure described above to embed W, and $\mathbb{L}$, into $C(\Omega)$.

According to the physical interpretation of the delinearization procedure we want to see $\Omega = \overline{\mathrm{Ex}S}^{W^*}$ as the phase space of a classical statistical theory. The basic elements of this theory are then $M(\Omega)$, the base normed Banach space of signed Radon measures on Ω, and $F(\Omega)$, the order-unit normed Banach space of measurable functions on Ω (compare Singer and Stulpe, 1993). We will identify $C(\Omega)$ with the corresponding subspace of $F(\Omega)$; thus, the delinearization map D is the injection of W into $F(\Omega)$.

It is known that the Boolean algebra of measurable subsets of Ω is canonically represented by the extreme elements of $[o, e]_F$; thus, the logic $\mathbb{L}_c$ of the classical statistical theory based on $F(\Omega)$ and $M(\Omega)$ equals $\mathrm{Ex}[o, e]_F$ (compare Singer and Stulpe, 1993). It is easy to see that the delinearization map represents $\mathrm{Ex}[o, e]_W$ by (in general) nonextreme elements of $[o, e]_F$. This suggests that the classical statistical theory resulting from the delinearization of the original quantal theory has to be an operational theory, i.e., its set of elementary observables has to be the full set $[o, e]_F$ instead of $\mathrm{Ex}[o, e]_F$. Thus, we should devote some attention to the algebraic structure of $[o, e]_F$, the logic of the operational classical statistical theory [a similar study for the case of operational standard quantum theory can be found in Bugajski (1981)].

3. ALGEBRAIC STRUCTURE OF $[o, e]_F$

The set $[o, e]_F$ is the set of effects of the operational classical statistical theory based on the statistical duality $M(\Omega)$, $F(\Omega)$. The notion of observable introduced above can be easily adapted to this case. The minimal algebraic structure on $[o, e]_F$ that we need in order to define observables is the partial algebra $([o, e]_F, +, -)$ with pointwise addition of functions restricted to $[o, e]_F$: if $a(\alpha)+b(\alpha)\leq 1$ for all $\alpha\in\Omega$ with $a, b\in[o, e]_F$, then the pointwise sum $a+b$ belongs to $[o, e]_F$, and if $a(\alpha)-b(\alpha)\geq 0$ for all $\alpha\in\Omega$ with $a, b\in[o, e]_F$, then the pointwise difference $a-b$ is an element of $[o, e]_F$. The partial binary operations $+$, $-$ correspond to the quantum-logical orthoaddition and relative orthocomplementation, respectively. Let us note that this partial algebraic structure on $[o, e]_F$ is well motivated by the physical theory.

The basic partial algebra $([o, e]_F, +, -)$ generates a rich collection of relations and operations on $[o, e]_F$ definable in terms of the two operations $+$, $-$. Thus the domain of the partial operation "$-$" determines a binary relation on $[o, e]_F$ which is evidently a partial order: $a\leq b$ iff $b-a$ is defined. The domain of the partial operation "$+$" determines another binary relation on $[o, e]_F$: $a\perp b$ iff $a+b$ is defined. The set $\{b|b\in[o, e]_F, b\perp a\}$ for any fixed $a\in[o, e]_F$ possesses the largest element with respect to $\leq$, which will be denoted $\sim a$ and called the quasicomplement of a.

The poset $([o, e]_F, \leq)$ is actually a distributive lattice that comes from the known lattice properties of $F(\Omega)$. For any $a, b\in[o, e]_F$ their lattice meet and lattice join are denoted by $a\wedge b$ and $a\vee b$, respectively. It is easy to see that $(a\wedge b)(\alpha)=\min\{a(\alpha), b(\alpha)\}$ and $(a\vee b)(\alpha)=\max\{a(\alpha), b(\alpha)\}$ for any $\alpha\in\Omega$. The constant-1 function e is the maximal element of $([o, e]_F, \leq)$, whereas the constant-0 function o is the minimal one. The quasicomplement $\sim a$ can be now equivalently described as $e-a$. The algebra $([o, e]_F, \wedge, \sim)$ with the order defined by $a\leq b$ iff $a\wedge b=a$, and the lattice join by $a\vee b=\sim(\sim a\wedge\sim b)$, is a quasi-Boolean algebra according to Rasiowa (1974). It should be stressed that, contrary to the properties of the traditional quantum orthomodular orthoposet, the sum $a+b$ does not coincide with the lattice join $a\vee b$ (nevertheless, $a+b\geq a\vee b$ for any $a, b\in[o, e]_F$ such that $a\perp b$), and the difference $a-b$ does not equal $\sim b\wedge a$. Evidently, $([o, e]_F, +, -, \sim)$ is not a partial Boolean algebra.

The distributive lattice $([o, e]_F, \leq, \wedge, \vee)$ is relatively pseudocomplemented, i.e., for any $a, b\in[o, e]_F$ there exists an element of $[o, e]_F$, denoted $a\hookrightarrow b$, such that for any $c\in[o, e]_F$, $a\wedge c\leq b$ iff $c\leq a\hookrightarrow b$. Indeed, $a\hookrightarrow b$ is defined by $(a\hookrightarrow b)(\alpha)=1$ iff $a(\alpha)\leq b(\alpha)$, and $(a\hookrightarrow b)(\alpha)=b(\alpha)$ otherwise. Owing to the presence of the minimal element o in $[o, e]_F$, we can define the pseudocomplementation $a\to\rceil a$ in $[o, e]_F$ by $\rceil a:=a\hookrightarrow o$. Thus, $([o, e]_F, \wedge, \vee, \hookrightarrow)$ with the order defined by $a\leq b$ iff $a\hookrightarrow b=e$, and the pseudocomplementation $a\to\rceil a=a\hookrightarrow o$, is a pseudo-Boolean algebra (alias Heyting algebra, or pseudocomplemented lattice) (Rasiowa, 1974). Thus,

$[o, e]_F$ is simultaneously a quasi-Boolean algebra and a pseudo-Boolean algebra with the same order relation and distributive lattice operations for both structures. It should be noted, however, that $[o, e]_F$ with its quasi-Boolean and pseudo-Boolean structures is not a quasi-pseudo-Boolean algebra as defined in Rasiowa (1974).

Among operations on $[o, e]_F$ which could be defined by combining the ones listed above, an especially interesting one is the unary operation: $a \to la := \rceil \sim a$. It is easy to check that l is an interior operation on the quasi-Boolean algebra $([o, e]_F, \vee, \sim)$, i.e., the following defining equalities (Rasiowa, 1974) hold for any $a, b \in [o, e]_F$: $l(a \wedge b) = la \wedge lb$, $la \wedge a = la$, $lla = la$, $le = e$. Thus $([o, e]_F, \wedge, \sim, l)$ could be called a "topological quasi-Boolean algebra" in analogy to the "topological Boolean algebra" of Rasiowa (1974). The operation $a \to ma := \sim l \sim a$ is a closure operation in $([o, e]_F, \vee, \sim)$. Observe that $\rceil a$ can then be described as the interior of the quasicomplement of a: $\rceil a = l \sim a$.

The Boolean algebra $\mathbb{B}(\Omega)$ of measurable subsets of Ω is canonically injected into $[o, e]_F$ by attaching to any measurable subset its characteristic function. Identifying $\mathbb{B}(\Omega)$ with its image under this injection, we see that $\mathbb{B}(\Omega)$ is exactly the set of extreme elements of the convex set $[o, e]_F$. The injection of $\mathbb{B}(\Omega)$ into $[o, e]_F$ is a Boolean monomorphism, i.e., it preserves the Boolean operations on measurable subsets of Ω. The relative pseudocomplementation can be restricted to $\mathbb{B}(\Omega)$ and coincides there with the standard operation: $\mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \ni (a, b) \to \sim a \vee b$. The interior and the closure operations both are surjections of $[o, e]_F$ onto $\mathbb{B}(\Omega)$ and both act trivially on $\mathbb{B}(\Omega)$: $la = a$, $ma = a$, for any $a \in \mathbb{B}(\Omega)$. The $\mathbb{B}(\Omega)$ is the only maximal Boolean sublattice of the quasi-Boolean and pseudo-Boolean algebra over $[o, e]_F$.

The relation of compatibility, crucial for all studies of quantum logic, can be defined for $[o, e]_F$ as well: two elements of $[o, e]_F$ are called compatible iff they both belong to the range of an observable. It appears that this relation is trivial for the nonlinear extension of any quantum theory, because any two elementary nonlinear observables (i.e., any two elements of $[o, e]_F$) are compatible as a consequence of the formula $a(\alpha) + b(\alpha) - (a \wedge b)(\alpha) = (a \vee b)(\alpha)$, which holds for any pair $a, b \in [o, e]_F$ and for all $\alpha \in \Omega$.

Let us summarize the above considerations. The delinearized quantum theory (or the nonlinear extension of the original quantum theory) is an operational classical statistical theory based on the phase space Ω introduced in Section 2. The set of effects (elementary observables) $[o, e]_F$ of the operational classical statistical theory is the set of elements of the logic of this theory. The logic of the operational classical statistical theory is the algebraic structure $([o, e]_F, +, -, \wedge, \hookrightarrow)$ with the basic operations described above and with a rich collection of derived relations and operations such as $\perp, \leq, \vee, \sim,], l, m$ introduced above.

4. ALGEBRAIC PROPERTIES OF D

It is easy to see that the injection $D\colon \mathbb{L} \to [o, e]_F$ has the following properties (for any $a, b \in \mathbb{L}$):

(i) $D(o) = o$, $D(e) = e$.
(ii) $a \leq b$ iff $D(a) \leq D(b)$.
(iii) $a \perp b$ iff $D(a) \perp D(b)$.
(iv) $D(a \vee b) = D(a) + D(b)$ for $a \perp b$.
(v) $D(\sim a \wedge b) = D(b) - D(a)$ for $a \leq b$.
(vi) $D(\sim a) = \sim D(a)$.

The listed properties suggest that a natural way to think about D is to consider it as a mapping between partial algebras. So the original orthomodular orthoposet $(\mathbb{L}, \leq, e, \sim)$ should be considered as a partial algebra $(\mathbb{L}, e, +, -)$ with distinguished element e and two partial binary operations $+, -$ defined as follows: for any $a, b \in \mathbb{L}$ such that $\alpha(a) + \alpha(b) \leq 1$ for all $\alpha \in \Omega$ there exists an element of $\mathbb{L}$, denoted $a + b$, such that $\alpha(a+b) = \alpha(a) + \alpha(b)$ for all $\alpha \in \Omega$; and for any $a, b \in \mathbb{L}$ such that $\alpha(a) - \alpha(b) \geq 0$ for all $\alpha \in \Omega$ there exists an element of $\mathbb{L}$, denoted $a - b$, such that $\alpha(a-b) = \alpha(a) - \alpha(b)$ for all $\alpha \in \Omega$. Obviously, $a + b$ is the orthogonal sum, denoted $a \vee b$ in the theory of orthoposets, and $a - b$ is the relative orthocomplement, denoted $\sim b \wedge a$ there. The relations $\leq, \perp$, as well as the unary operation $\sim$, can be derived in the partial algebra $(\mathbb{L}, e, +, -)$ in the same way as in the case of $[o, e]_F$ of Section 3.

Now we can say that the delinearization map $D\colon \mathbb{L} \to [o, e]_F$ is a monomorphism of the partial algebra $(\mathbb{L}, e, +, -)$ into the partial algebra $([o, e]_F, +, -)$ which consequently preserves the derived relations $\leq$ and $\perp$, as well as the unary operation $\sim$. This is the content of properties (i)–(vi). This result should be expected, as all the observables of the original quantum theory should also be observables of the delinearized theory. The last condition seems to be a necessary condition for the nonlinear theory to be a genuine extension of the original one.

Let us observe that the delinearization completely destroys the lattice properties of $(\mathbb{L}, \leq)$. The lattice joins and meets eventually existing in the poset $(\mathbb{L}, \leq)$, including the joins and meets of orthogonal elements, are not preserved under D (except for trivial cases like $a \vee e$, etc.). Thus, the lattice properties of the original quantum logic become unessential when we pass to the nonlinear extension. The same concerns an eventual atomicity of $(\mathbb{L}, \leq)$: the lattice $([o, e]_F, \wedge, \vee)$ is atomic; nevertheless, D maps atoms of $\mathbb{L}$

into nonatomic elements of $[o, e]_F$ (except for pathological cases of extremely restrictive superselection rules).

It should also be noted that D does not map elements of $\mathbb{L}$ on extreme elements of $[o, e]_F$ (except for elements of the center of $\mathbb{L}$); thus, the original elementary observables are not represented in general by the measurable subsets of Ω. This is the reason why we have to take into account the full set $[o, e]_F$ rather than its subset $\mathbb{B}(\Omega)$ as the set of all elementary observables of the delinearized theory.

Applying the interior operation l after D, we get $D_l := lD$, which maps $\mathbb{L}$ into $\mathbb{B}(\Omega) \subset [o, e]_F$. Generally, D_l is not an injection, which can be demonstrated by simple examples [such as the one described in Davies (1972)], but it happens that in the special case of standard quantum mechanics D_l is one-to-one. The mapping D_l corresponds to the Boolean embedding of quantum logic constructed by Beltrametti and Cassinelli (1976). It preserves only the order and both the maximal and the minimal elements of $\mathbb{L}$. Similar properties show $D_0 := mD$. A discouraging feature of D_l and D_0 is that they do not preserve the partial operations $+$, $-$ on $\mathbb{L}$, so the original observables do not survive the transformation of $\mathbb{L}$ into $\mathbb{B}(\Omega)$ done by D_l or D_0.

5. INNER LANGUAGE OF THE DELINEARIZED QUANTUM THEORY

It is well established that physical theories (at least some of them) contain semantic and logical structures, classical ones only in the case of standard classical mechanics. One of the ways of extracting the inner language of a physical theory, following from the work of Birkhoff, von Neumann, and Finkelstein, is described in Bugajski (1982). The considerations of the preceding sections make it possible to apply the extraction procedure of Bugajski (1982) to the case of the operational classical statistical theory (OCT for short) appearing as the result of the delinearization of a quantal theory.

According to Bugajski (1982), we identify $[o, e]_F$ as the set of all propositions (in the semantic sense) of the inner language of OCT. The algebra $([o, e]_F, +, -, \wedge, \hookrightarrow)$ should be seen now as the semantic algebra of IL(OCT)—the inner language of OCT. The set Ω is to be identified with the set of possible worlds of the arising semantics, or with its valuation space (compare van Fraassen, 1971). The obtained semantic structure provides an example of the Banach-space semantics as described in Bugajski (1983).

The algebraic operations on $[o, e]_F$ should be considered as semantic counterparts of sentential connectives of IL(OCT). Some of them show properties similar to those of sentential connectives of known formal languages. Thus, e.g., the lattice operations $\wedge$, $\vee$ over $[o, e]_F$ should correspond to disjunction and conjunction, respectively, whereas $\sim$ corresponds to a kind of negation. The "topological" operations l, m are to be interpreted as semantic images of modal operators of necessity and possibility, respectively (Rasiowa, 1974). The operation $\rceil = l\sim$ represents a kind of "strict negation." The pseudo-Boolean structure of $[o, e]_F$ shows that IL(OCT) contains a fragment syntactically identical with a language of intuitionistic logic (Rasiowa, 1974).

The delinearization map defines a translation of the inner language of the original quantal theory IL(OQT) into IL(OCT). The translation does not change the valuation space Ω, but the algebraic properties of D show that the conjunctions and disjunctions of IL(OQT) are not preserved under the translation, whereas the negation of IL(OQT) is translated into the "quasinegation" of IL(OCT).

6. RELATION TO HIDDEN VARIABLES

The theory resulting from the delinearization of a quantal theory is a genuine classical theory in its operational generalization, which seemingly contradicts anti-hidden variables theorems. All such theorems refer, however, to a Boolean embedding of the original quantum logic. Physical intuitions behind it imagine $\mathbb{L}$ as a subset of the Boolean algebra of measurable subsets of a phase space of a classical statistical theory. The successful realization of the hidden variables idea should then provide a mapping which transforms sharp quantal observables into sharp classical observables. Such a map is known to be impossible, which has been demonstrated in many ways. A typical result of this kind is: There is no injection H of $(\mathbb{L}, +, -)$ into a Boolean lattice $(\mathbb{B}, e, \wedge, \sim)$ such that $H(e)=e$, $H(a+b)=H(a) \vee H(b)$ for all, a, b such that $a+b$ is defined, $H(a-b)=H(a) \wedge H(\sim b)$ for all a, b such that $a-b$ is defined (see, e.g., Zierler and Schlesinger, 1965). Our map D meets all these conditions simply because it maps $\mathbb{L}$ into the non-Boolean algebraic structure of $[o, e]_F$ described above, which is the logic of classical, nevertheless operational, statistical theory. The root of all the impossibility theorems for classical embeddings of quantal theories is the unnecessarily restrictive condition demanding that quantal sharp observables should be represented by classical sharp observables.

On the other hand, it seems that our delinearization procedure does not fit well the idea of hidden variables. The adherents of the hidden variables reinterpretation of quantum mechanics want to explain quantal probabilities as resulting from classical statistics, so they imagine quantal pure states as nontrivial probability measures on a phase space of the hypothetical hidden variables theory. Such a representation of quantal states appears possible, but it does not provide any Boolean embedding of the quantum logic [for a short review and summary see Bugajski (1991)]. From the point of view of the delinearization procedure, the situation is somehow opposed: the classical embedding of a quantum theory provided by the Kadison map shows that all specific quantal peculiarities result from restricting the set of observables of a classical operational theory.

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